LAST PASSAGE PERCOLATION AND TRAVELING FRONTS

FRANCIS COMETS[1,4], JEREMY QUASTEL[2] AND ALEJANDRO F. RAMÍREZ[3,4]

Abstract. We consider a system of N particles with a stochastic dynamics introduced by Brunet and Derrida [7]. The particles can be interpreted as last passage times in directed percolation on $\{1,\ldots,N\}$ of mean-field type. The particles remain grouped and move like a traveling front, subject to discretization and driven by a random noise. As N increases, we obtain estimates for the speed of the front and its profile, for different laws of the driving noise. As shown in [7], the model with Gumbel distributed jumps has a simple structure. We establish that the scaling limit is a Lévy process in this case. We study other jump distributions. We prove a result showing that the limit for large N is stable under small perturbations of the Gumbel. In the opposite case of bounded jumps, a completely different behavior is found, where finite-size corrections are extremely small.

1. Definition of the model

We consider the following stochastic process introduced by Brunet and Derrida [7]. It consists in a fixed number $N \geq 1$ of particles on the real line, initially at the positions $X_1(0),\ldots,X_N(0)$. With $\{\xi_{i,j}(s) : 1 \leq i,j \leq N, s \geq 1\}$ an i.i.d. family of real random variables, the positions evolve as

$$X_i(t+1) = \max_{1\leq j\leq N} \big\{ X_j(t) + \xi_{i,j}(t+1) \big\}. \tag{1.1}$$

The components of the N-vector $X(t) = (X_i(t), 1 \leq i \leq N)$ are not ordered. The vector $X(t)$ describes the location after the t-th step of a population under reproduction, mutation and selection keeping the size constant. Given the current positions of the population, the next positions are a N-sample of the maximum of the full set of previous ones evolved by an independent step. It can be also viewed as long-range directed polymer in random medium with N sites in the transverse direction,

$$X_i(t) = \max \Big\{ X_{j_0}(0) + \sum_{s=1}^{t} \xi_{j_s,j_{s-1}}(s) ; 1 \leq j_s \leq N \ \forall s = 0,\ldots t-1, j_t = i \Big\}, \tag{1.2}$$

as can be checked by induction ($1 \leq i \leq N$). The model is long-range since the maximum in (1.1) ranges over all j's. For comparison with a short-range model, taking the maximum over j neighbor of i in $\mathbb{Z}$ in (1.1) would define the standard oriented last passage percolation model with passage time ξ on edges in two dimensions.

Date: January 28, 2013.

AMS 2000 *subject classifications.* Primary 60K35, 82C22; secondary 60G70, 82B43.

Key words and phrases. Last passage percolation, Traveling wave, Interacting Particle Systems, Front propagation, Brunet-Derrida correction.

[1]Université Paris Diderot. Partially supported by CNRS, Laboratoire de Probabilités et Modèles Aléatoires, UMR 7599.

[2]University of Toronto. Partially supported by NSERC, Canada.

[3]Pontificia Universidad Católica de Chile. Partially supported by Fondo Nacional de Desarrollo Científico y Tecnológico grant 1100298.

[4]Partially supported by ECOS-Conicyt grant CO9EO5.

By the selection mechanism, the N particles remain grouped even when $N \to \infty$, they are essentially pulled by the leading ones, and the global motion is similar to a front propagation in reaction-diffusion equations with traveling waves. Two ingredients of major interest are: (i) the discretization effect of a finite N, (ii) the presence of a random noise in the evolution. Such fronts are of interest, but poorly understood; see [26] for a survey from a physics perspective.

Traveling fronts appear in mean-field models for random growth. This was discovered by Derrida and Spohn [13] for directed polymers in random medium on the tree, and then extended to other problems [22, 23].

The present model was introduced by Brunet and Derrida in [7] to compute the corrections for large but finite system size to some continuous limit equations in front propagation. Corrections are due to finite size, quantization and stochastic effects. They predicted, for a large class of such models where the front is pulled by the farmost particles [7, 8], that the motion and the particle structure have universal features, depending on just a few parameters related to the upper tails. Some of these predictions have been rigorously proved in specific contexts, such as the corrections to the speed of the Branching Random Walk (BRW) under the effect of a selection [4], of the solution to KPP equation with a small stochastic noise [24], or the genealogy of branching Brownian motions with selection [3]. For the so-called N-BBM (branching Brownian motion with killing of leftmost particles to keep the population size constant and equal to N) the renormalized fluctuations for the position of the killing barrier converge to a Levy process as N diverges [21].

We mention other related references. For a continuous-time model with mutation and selection conserving the total mass, the empirical measure converges to a free boundary problem with a convolution kernel [15]. Traveling waves are given by a Wiener-Hopf equation. For a different model mimicking competition between infinitely many competitors, called Indy-500, quasi-stationary probability measures for competing particles seen from the leading edge corresponds to a superposition of Poisson processes [28]. For diffusions interacting through their rank, the spacings are tight [27], and the self-normalized exponential converge to a Poisson-Dirichlet law [10]. In [1], particles jump forward at a rate depending on their relative position with respect to the center of mass, with a higher rate for the particle behind: convergence to a traveling front is proved, which is given in some cases by the Gumbel distribution.

We now give a flavor of our results. The Gumbel law $\mathrm{G}(0,1)$ has distribution function $\mathbb{P}(\xi \leq x) = \exp -e^{-x}, x \in \mathbb{R}$. In [7] it is shown that an appropriate measure of the front location of a state $X \in \mathbb{R}^N$ in this case is

$$\Phi(X) = \ln \sum_{1 \leq j \leq N} e^{X_j} ,$$

and that $\Phi(X(t))$ is a random walk, a feature which simplifies the analysis. For an arbitrary distribution of ξ, the speed of the front with N particles can be defined as the almost sure limit

$$v_N = \lim_{t \to \infty} t^{-1} \Phi(X(t)) .$$

We emphasize that N is fixed in the previous formula, though it is sent to infinity in the next result. Our first result is the scaling limit as the number N of particles diverges.

Theorem 1.1. *Assume $\xi_{i,j}(t) \sim \mathrm{G}(0,1)$. Then, for all sequences $m_N \to \infty$ as $N \to \infty$,*

$$\frac{\Phi(X([m_N \tau])) - \beta_N m_N \tau}{m_N / \ln N} \xrightarrow{\text{law}} \mathcal{S}(\tau)$$

in the Skorohod topology with $\mathcal{S}(\cdot)$ a totally asymmetric Cauchy process with Lévy exponent ψ_C from (3.27), where

$$\beta_N = \ln b_N + N b_N^{-1} \ln m_N,$$

with $\ln b_N = \ln N + \ln\ln N - \frac{\gamma}{\ln N} + \mathcal{O}(\frac{1}{\ln^2 N})$, see (3.28).

Fluctuations of the front location are Cauchy distributed in the large N limit. Keeping N fixed, the authors in [7] find that they are asymptotically Gaussian as $t \to \infty$. We prove here that, as N is sent to infinity, they are stable with index 1, a fact which has been overlooked in [7]. When large populations are considered, this is the relevant point of view. The Cauchy limit also holds true in the boundary case when time is not speeded-up ($m_N = 1$) and $N \to \infty$. For most growth models, finding the scaling limit is notoriously difficult. In the present model, it is not difficult for the Gumbel distribution, but remains an open question for any other distribution.

We next consider the case when ξ is a perturbation of the Gumbel law. Define $\varepsilon(x) \in [-\infty, 1]$ by

$$\varepsilon(x) = 1 + e^x \ln \mathbb{P}(\xi \le x). \tag{1.3}$$

Note that $\varepsilon \equiv 0$ is the case of $\xi \sim \mathrm{G}(0,1)$. The empirical distribution function (more precisely, its complement to 1) of the N-particle system (1.1) is the random function

$$U_N(t,x) = N^{-1} \sum_{i=1}^{N} \mathbf{1}_{X_i(t)>x} \tag{1.4}$$

This is a non-increasing step function with jumps of size $1/N$ and limits $U_N(t,-\infty) = 1$, $U_N(t,+\infty) = 0$. It has the shape of a front wave, propagating at mean speed v_N, and it combines two interesting aspects: randomness and discrete values. We will call it the front profile, and we study in the next result its relevant part, around the front location.

Theorem 1.2. *Assume that*

$$\lim_{x\to+\infty} \varepsilon(x) = 0, \qquad \text{and} \quad \varepsilon(x) \in [-\delta^{-1}, 1-\delta], \tag{1.5}$$

for all x and some $\delta > 0$. Then, for all initial configurations $X(0) \in \mathbb{R}^N$, all $k \ge 1$, all $K_N \subset \{1,\dots,N\}$ with cardinality k, and all $t \ge 2$ we have

$$\Big(X_j(t) - \Phi(X(t-1)); j \in K_N \Big) \xrightarrow{\text{law}} G(0,1)^{\otimes k}, \quad N \to \infty, \tag{1.6}$$

with Φ from (3.20), and moreover,

$$U_N\big(t, \Phi(X(t-1)) + x\big) \longrightarrow u(x) = 1 - e^{-e^{-x}} \tag{1.7}$$

uniformly in probability as $N \to \infty$.

As is well known, it is rare to find rigorous perturbation results from exact computations for such models. For example, the above mentioned, last passage oriented percolation model on the planar lattice, is exactly sovable for exponential passage times [19] or geometric ones [2] on sites, and the fluctuations asympotically have a Tracy-Widom distribution. However, no perturbative result has been obtained after a decade. Even though our assumptions seem to be strong, it is somewhat surprising that we can prove this result. The second condition is equivalent to the following stochastic domination: there exist finite constants $c < d$ ($c = \ln \delta, d = \ln(1+\delta^{-1})$) such that

$$g + c \le_{\text{sto}} \xi \le_{\text{sto}} g + d, \qquad g \sim G(0,1). \tag{1.8}$$

This condition is reminiscent of assumption (1.13) in [25] used to control the fluctuations of the front location for KPP equation in random medium. By Theorem 1.2, as $N \to \infty$, the front

remains sharp and its profile, which is defined microscopically as the empirical distribution function of particles, converges to the Gumbel distribution as $N \to \infty$. Hence the Gumbel distribution is not only stable, but it is also an attractor.

Finally, we study the finite-size corrections to the front speed in a case when the distribution of ξ is quite different from the Gumbel law.

Theorem 1.3. *Let $b < a$ and $p \in (0,1)$, and assume that the $\xi_{i,j}(t)$'s are integrable and satisfy*

$$\mathbb{P}(\xi > a) = \mathbb{P}(\xi \in (b,a)) = 0, \qquad \mathbb{P}(\xi = a) = p, \qquad \mathbb{P}(\xi \in (b-\varepsilon, b]) > 0 \tag{1.9}$$

for all $\varepsilon > 0$. Then, as $N \to \infty$,

$$v_N = a - (a-b)(1-p)^{N^2} 2^N + o\big((1-p)^{N^2} 2^N\big).$$

We note that in such a case, in the leading order terms of the expansion as $N \to \infty$, the value of the speed depends only on a few features of the distribution of ξ: the largest value a, its probability mass p and the gap $a-b$ with second largest one. All these involve the top of the support of the distribution, the other details being irrelevant. Such a behavior is expected for pulled fronts.

Though the mechanisms are different, we make a parallel between the model considered here, and the BRW with selection, in order to discuss the Brunet-Derrida correction of the front speed v_N with respect to its asymptotic value. For definiteness, denote by η the displacement variable, assume that η is a.s. bounded from above by a constant a, and assume the branching is constant and equal to $\beta > 1$. The results of [4] are obtained for $\beta \times \mathbb{P}(\eta = a) < 1$ (Assumption A3 together with Lemma 5 (3) in [5]), resulting in a logarithmic correction: This case corresponds to the Gumbel distribution for ξ in our model, e.g., to Theorems 1.1 and 1.2 . In contrast, the assumptions of Theorem 1.3 yield a much smaller correction (of order exponential of negative N^2). This other case corresponds for large N to the assumption $\beta \times \mathbb{P}(\eta = a) > 1$ for the BRW with selection, where the corrections are exponentially small [11], precisely given by ρ^N with $\rho < 1$ the extinction probability of the supercritical Galton-Watson process of particles located at site ta at time t. In our model, the branching number is N and ρ is itself exponentially small, yielding the correct exponent of negative N^2, but not the factor 2^N.

The paper is organised as follows. Section 2 contains some standard facts for the model. Section 3 deals with the front location in the case of the Gumbel law for ξ. In Section 4, we study the asymptotics as $N \to \infty$ of the front profile (for Gumbel law and small perturbations), and their relations to traveling waves and reaction-diffusion equation. In Sections 5 and 6, we expand the speed in the case of integer valued, bounded from above, ξ's, starting with the Bernoulli case. Theorems 1.1 and 1.3 are proved in Sections 3.3 and 6.3 respectively.

2. Preliminaries for fixed N

For any fixed N, we show here the existence of large time asymptotics for the N-particles system. It is convenient to shift the whole system by the position of the leading particle, because we show that there exists an invariant measure for the shifted process.

The ordered process: We now consider the process $\tilde{X} = (\tilde{X}(t), t \in \mathbb{N})$ obtained by ordering the components of $X(t)$ at each time t, i.e., the set $\{\tilde{X}_1(t), \tilde{X}_2(t), \dots, \tilde{X}_N(t)\}$ coincides with $\{X_1(t), X_2(t), \dots, X_N(t)\}$ and $\tilde{X}_1(t) \geq \tilde{X}_2(t) \geq \cdots \geq \tilde{X}_N(t)$. Then, $\tilde{X}$ is a Markov chain with state space

$$\Delta_N := \{y \in \mathbb{R}^N : y_1 \geq y_2 \geq \ldots \geq y_N\}.$$

Given $\tilde{X}(t)$, the vector $X(t)$ is uniformly distributed on the $N!$ permutations of $\tilde{X}(t)$. Hence, it is sufficient to study $\tilde{X}$ instead of X. It is easy to see that the sequence $\tilde{X}$ has the same law as $Y = (Y(t), t \geq 0)$, given by as a recursive sequence

$$Y(t+1) = \text{ordered vector}\Big(\max_{1\leq j\leq N} \big\{Y_j(t) + \xi_{i,j}(t+1)\big\}, 1 \leq i \leq N\Big). \tag{2.10}$$

Note that, when $X(0)$ is not ordered, $\tilde{X}(1)$ is not a.s. equal to $Y(1)$ starting from $Y(0) = \tilde{X}(0)$. In this section we study the sequence Y, which is nicer than $\tilde{X}$ because of the recursion (2.10): Denote by $T_{\xi(t+1)}$ the above mapping $Y(t) \mapsto Y(t+1)$ on Δ_N, and observe first that

$$Y(t) = T_{\xi(t)} \dots T_{\xi(2)} T_{\xi(1)} Y(0). \tag{2.11}$$

For $y, x \in \Delta_N$, write $y \leq x$ if $y_i \leq x_i$ for all $i \leq N$. The mapping $T_{\xi(t)}$ is monotone for the partial order on Δ_N, i.e., for the solutions Y, Y' of (2.10) starting from $Y(0), Y'(0)$ we have

$$Y(0) \leq Y'(0) \Longrightarrow Y(t) \leq Y'(t),$$

and moreover, with $\mathbf{1} = (1, 1, \dots, 1)$, $r \in \mathbb{R}$ and $y \in \mathbb{R}^N$,

$$T_{\xi(t)}(y + r\mathbf{1}) = r\mathbf{1} + T_{\xi(t)}(y). \tag{2.12}$$

The process seen from the leading edge: For each $x \in \mathbb{R}^N$, we consider its shift x^0 by the maximum,

$$x_i^0 = x_i - \max_{1\leq j\leq N} x_j,$$

and the corresponding processes X^0, Y^0. We call X^0, Y^0, the unordered process, respectively, the ordered process, seen from the leading edge. Note that $T_{\xi(t)}(y^0) = T_{\xi(t)}(y) - (\max_j y_j)\mathbf{1}$ by (2.12), which yields

$$\Big(T_{\xi(t)}(y^0)\Big)^0 = \Big(T_{\xi(t)}(y)\Big)^0;$$

a similar relation holds for x's instead of y's. Then X^0, Y^0 are Markov chains, with Y^0 taking values in $\Delta_N^0 := \{y \in \Delta_N : y_1 = 0\}$, and we denote by ν_t the law of $Y^0(t)$.

Proposition 2.1. *There exists an unique invariant measure ν for the process Y^0 seen from the leading edge, and we have*

$$\lim_{t\to\infty} \nu_t = \nu. \tag{2.13}$$

Furthermore, there exists a $\delta_N > 0$ such that

$$||\nu_t - \nu||_{TV} \leq (1 - \delta_N)^t. \tag{2.14}$$

Similar results hold for the unordered process X^0, by the remark preceeding (2.10). Also, we mention that the value of δ_N is not sharp.

Proof. Consider the random variable

$$\tau = \inf \big\{t \geq 1 : \xi_{i,1}(t) = \max\{\xi_{i,j}(t); j \leq N\} \ \forall i \leq N\big\}.$$

Then, τ is a stopping time for the filtration $(\mathcal{F}_t)_{t\geq 0}$, with $\mathcal{F}_t = \sigma\{\xi_{i,j}(s); s \leq t, i, j \geq 1\}$. It is geometrically distributed with parameter not smaller than

$$\delta_N = (1/N)^N. \tag{2.15}$$

Denote by $\oplus, \ominus$ the configuration vectors

$$\oplus = (0, 0, \dots, 0), \qquad \ominus = (0, -\infty, \dots, -\infty).$$

They are extremal configurations in (the completion of) Δ_N^0 since $\ominus \le y \le \oplus$ for all $y \in \Delta_N^0$. Now, by definition of τ and (2.10),

$$T_{\xi(\tau)}\oplus = T_{\xi(\tau)}\ominus = T_{\xi(\tau)}y \qquad \forall y \in \Delta_N^0.$$

Hence, for all $t \ge \tau$ and all $y \in \Delta_N$ such that $\max_{1\le j\le N} y_j = \max_{1\le j\le N} Y_j(0)$,

$$Y(t) = T_{\xi(t)} \dots T_{\xi(2)} T_{\xi(1)} y.$$

We can construct a renewal structure. Define $\tau_1 = \tau$, and recursively for $k \ge 0$, $\tau_{k+1} = \tau_k + \tau \circ \theta_{\tau_k}$ with θ the time-shift. This sequence is the success time sequence in a Bernoulli process, we have $1 \le \tau_1 < \tau_2 < \dots < \tau_k < \dots < \infty$ a.s. The following observation is plain but fundamental.

Lemma 2.1 (Renewal structure)**.** *The sequence*

$$(Y^0(s); 0 \le s < \tau_1), (Y^0(\tau_1+s); 0 \le s < \tau_2-\tau_1), (Y^0(\tau_2+s); 0 \le s < \tau_3-\tau_2), \dots$$

is independent. Moreover, for all $k \ge 1$, $(Y^0(\tau_k+s); s \ge 0)$ has the same law as $(Y^0(1+s); s \ge 0)$ starting from $Y^0(0) = \oplus$.

Proof. of Lemma 2.1. By the strong Markov property, the Markov chain Y^0 starts afresh from the stopping times $\tau_1 < \tau_2 < \dots$. This proves the first statement, and we now turn to the second one. Note that $T_\xi \oplus = T_\eta \oplus$ if, for all i, $(\xi_{i,j}; j \le N)$ is a permutation of $(\eta_{i,j}; j \le N)$. Hence,

$$\mathbb{P}\big(Y^0(1) \in \cdot\,, \tau_1 = 1\, |Y^0(0) = \oplus\big) = \mathbb{P}\big(Y^0(1) \in \cdot\, |Y^0(0) = \oplus\big) \times \mathbb{P}(\tau_1 = 1),$$

and so

$$\mathbb{P}\big(Y^0(1) \in \cdot\, |Y^0(0) = \oplus, \tau_1 = 1\big) = \mathbb{P}\big(Y^0(1) \in \cdot\, |Y^0(0) = \oplus\big).$$

From the markovian structure and by induction it follows that

$$\begin{aligned}
\mathbb{P}\big((Y^0(1+s); s \ge 0) \in \cdot\, |Y^0(0) = \oplus\big) &= \mathbb{P}\big((Y^0(1+s); s \ge 0) \in \cdot\, |Y^0(0) = \oplus, \tau_1 = 1\big) \\
&= \mathbb{P}\big((Y^0(1+s); s \ge 0) \in \cdot\, |Y^0(0) = z, \tau_1 = 1\big) \\
&= \mathbb{P}\big((Y^0(1+s); s \ge 0) \in \cdot\, |\tau_1 = 1\big) \\
&= \mathbb{P}\big((Y^0(\tau_1+s); s \ge 0) \in \cdot\big),
\end{aligned}$$

for all $z \in \Delta_N^0$. ∎

The lemma implies the proposition, with the law ν given for a measurable $F : \Delta_N^0 \to \mathbb{R}_+$ by

$$\begin{aligned}
\int F d\nu &= \frac{1}{\mathbb{E}(\tau_2 - \tau_1)} \mathbb{E} \sum_{\tau_1 \le t < \tau_2} F(Y^0(t)) \\
&= \frac{1}{\mathbb{E}(\tau_1)} \sum_{t \ge 1} \mathbb{E}\big(F(Y^0(t)) \mathbf{1}_{t<\tau_2} | \tau_1 = 1\big). \qquad (2.16)
\end{aligned}$$

∎

Remark 2.2. *(i) The proposition shows that the particles remain grouped as t increases, i.e., the law of the distance between extreme particles is a tight sequence under the time evolution. In Theorem 1.2 we will see that when the law of ξ is close to Gumbel, they remain grouped too as N increases.*

(ii) The location of front at time t can be described by any numerical function $\Phi(Y(t))$ or $\Phi(Y(t-1))$ (or equivalently, any symmetric function of $X(t)$ or $X(t-1)$) which commutes to space translations by constant vectors,

$$\Phi(y + r\mathbf{1}) = r + \Phi(y) \ , \tag{2.17}$$

and which is increasing for the partial order on $\mathbb{R}^N$. Among such, we mention also the maximum or the minimum value, the arithmetic mean, the median or any other order statistics, and the choice in (3.20) below. For Proposition 2.1, we have taken the first choice – the location of the rightmost particle – for simplicity. Some other choices may be more appropriate to describe the front, by looking in the bulk of the system rather than at the leading edge. For fixed N all such choices will however lead to the same value for the speed v_N of the front, that we define below.

Note that for a function Φ which satisfies the commutation relation (2.17) we have the inequalities

$$\Phi(\oplus) + \min_{i\le N}\max_{j\le N}\{Y_j(0)+\xi_{i,j}(1)\} \le \Phi(Y(1)) \le \Phi(\oplus) + \max_{i,j\le N}\{Y_j(0)+\xi_{i,j}(1)\} \, .$$

Now, by equation (2.16) and by the fact that τ_1 is stochastically smaller than a geometric random variable with parameter $(1/N)^N$ we conclude that if $\xi \in L^p, Y(0) \in L^p$ then $\Phi(Y(t)) \in L^p$, and also $\int |y_N|^p d\nu(y) < \infty$. The following corollary is a straightforward consequence of the above.

Corollary 2.1 (Speed of the front)**.** *If $\xi \in L^1$, the following limits*

$$v_N = \lim_{t\to\infty} t^{-1}\max\{X_i(t); 1 \le i \le N\} = \lim_{t\to\infty} t^{-1}\min\{X_i(t); 1 \le i \le N\}$$

exist a.s., and v_N is given by

$$v_N = \int_{\Delta_N^0} d\nu(y)\, \mathbb{E} \max_{1\le i,j\le N} \big\{y_j + \xi_{i,j}(1)\big\}.$$

Moreover, if $\xi \in L^2$,

$$t^{-1/2}\big(\max\{X_i(t); 1 \le i \le N\} - v_N t\big)$$

converges in law as $t \to \infty$ to a Gaussian r.v. with variance $\sigma_N^2 \in (0,\infty)$.

We call v_N the speed of the front of the N-particle system.

Proof. The equality of the two limits in the definition of v_N follows from tightness in Remark 2.2, (i), and the existence is from the renewal structure. Similarly, we have

$$v_N = \int_{\Delta_N^0} \big(\mathbb{E}\Phi(T_\xi y) - \Phi(y)\big)\, d\nu(y)$$

for all Φ as in Remark 2.2, (ii), where Δ_N^0 is defined just before Proposition 2.1. The second formula is obtained by taking $\Phi(y) = \max_{i\le N} y_i$. The Gaussian limit is the Central Limit Theorem for renewal processes. ∎

3. The Gumbel distribution

The Gumbel law $\mathrm{G}(a,\lambda)$ with scaling parameter $\lambda > 0$ and location parameter $a \in \mathbb{R}$ is defined by its distribution function

$$\mathbb{P}(\xi \leq x) = \exp\big(-e^{-\lambda(x-a)}\big), \qquad x \in \mathbb{R}. \tag{3.18}$$

This law is known to be a limit law in extreme value theory [20]. In [7], Brunet and Derrida considered the standard case $a = 0, \lambda = 1$, to find a complete explicit solution to the model. In this section, we assume that the sequence $\xi_{i,j}$ is $\mathrm{G}(a,\lambda)$-distributed, for some a, λ. Then, $\zeta = \lambda(\xi - a) \sim \mathrm{G}(0,1)$, while

$$\exp(-\zeta) \quad \text{is exponentially distributed with parameter 1}, \tag{3.19}$$

and $\exp(-e^{-\zeta})$ is uniform on $(0,1)$. Conversely, if U is uniform on $(0,1)$ and $\mathcal{E}$ exponential of parameter 1, then $-\lambda \ln\ln(1/U)$ and $\ln \mathcal{E}^{-\lambda}$ are $G(0,\lambda)$.

Here, the Gumbel distribution makes the model stationary for fixed N and allows exact computations.

3.1. **The Front as a random walk.** In this section, we fix $N \geq 1, a \in \mathbb{R}, \lambda > 0$. We will choose the function $\Phi : \mathbb{R}^N \to \mathbb{R}$,

$$\Phi(x) = \lambda^{-1} \ln \sum_{i=1}^{N} \exp \lambda x_i \tag{3.20}$$

to describe the front location $\Phi(X(t))$ at time t.

Theorem 3.1 ([7]). *Assume the $\xi_{i,j}$'s are Gumbel $G(a,\lambda)$-distributed.*
(i) Then, the sequence $(\Phi(X(t)); t \geq 0)$ is a random walk, with increments

$$\Upsilon = a + \lambda^{-1} \ln \left(\sum_{i=1}^{N} \mathcal{E}_i^{-1} \right) \tag{3.21}$$

where the $\mathcal{E}_i$ are i.i.d. exponential of parameter 1.
(ii) Then,

$$v_N = a + \lambda^{-1} \mathbb{E} \ln \left(\sum_{i=1}^{N} \mathcal{E}_i^{-1} \right), \qquad \sigma_N^2 = \lambda^{-2} \mathrm{Var} \left(\ln \sum_{i=1}^{N} \mathcal{E}_i^{-1} \right). \tag{3.22}$$

(iii) The law ν from proposition 2.1 is the law of the shift $V^0 \in \Delta_N^0$ of the ordered vector V obtained from a N-sample from a Gumbel $G(0,\lambda)$.

Proof. : Define $\mathcal{F}_t = \sigma(\xi_{i,j}(s), s \leq t, i, j \leq N)$, and $\mathcal{E}_{i,j}(t) = \exp\{-\lambda(\xi_{i,j}(t) - a)\}$. By (1.1),

$$\begin{aligned} X_i(t+1) &= \max_{1 \leq j \leq N} \big\{ X_j(t) + a - \lambda^{-1} \ln \mathcal{E}_{i,j}(t+1) \big\} \\ &= a + \Phi(X(t)) - \lambda^{-1} \ln \mathcal{E}_i(t+1), \end{aligned} \tag{3.23}$$

where

$$\mathcal{E}_i(t+1) = \min_{1 \leq j \leq N} \big\{ \mathcal{E}_{i,j}(t+1) e^{-\lambda X_j(t)} \big\} e^{\lambda \Phi(X(t))}, \qquad t \geq 0.$$

Given $\mathcal{F}_t$, each variable $\mathcal{E}_i(t+1)$ is exponentially distributed with parameter 1 by the standard stability property of the exponential law under independent minimum, and moreover, the whole vector $(\mathcal{E}_i(t+1), i \leq N)$ is conditionnally independent. Therefore, this vector is independent of $\mathcal{F}_t$, and finally,

$$(\mathcal{E}_i(t), 1 \leq i \leq N, t \geq 1) \quad \text{is independent and identically distributed}$$

with parameter 1, exponential law. Hence, the sequence

$$\Upsilon(t) = a + \lambda^{-1} \ln \left(\sum_{i=1}^{N} \mathcal{E}_i(t)^{-1} \right), \qquad t \geq 1,$$

is i.i.d. with the same law as Υ. Now, by (3.23),

$$\begin{aligned} \Phi(X(t)) &= \Phi(X(t-1)) + \Upsilon(t) \\ &= \Phi(X(0)) + \sum_{s=1}^{t} \Upsilon(s) \end{aligned}$$

which shows that $(\Phi(X(t)); t \geq 0)$ is a random walk. Thus, we obtain both (i) and (ii).

From (3.23), we see that the conditional law of $X(t+1)$ given $\mathcal{F}_t$ is the law of a N-sample from a Gumbel $\mathrm{G}(a + \Phi(X(t)), \lambda)$. Hence, ν is the law of the order statistics of a N-sample from a Gumbel $\mathrm{G}(0, \lambda)$, shifted by the leading edge. ∎

We end this section with a remark. Observe that the other max-stable laws (Weibull and Frechet) do not yield exact computations for our model. Hence, the special role of the Gumbel is not due to the stability of that law under taking the maximum of i.i.d. sample, but also to its behavior under shifts.

3.2. **Asymptotics for large N.** In this section we study the asymptotics as $N \to \infty$ with a stable limit law. When $a = 0$ and $\lambda = 1$, Brunet and Derrida [7] obtain the expansions

$$v_N = \ln N + \ln \ln N + \frac{\ln \ln N}{\ln N} + \frac{1-\gamma}{\ln N} + o(\frac{1}{\ln N}), \tag{3.24}$$

$$\sigma_N^2 = \frac{\pi^2}{3 \ln N} + \ldots, \tag{3.25}$$

by Laplace method for an integral representation of the Laplace transform of Υ. We recover here the first terms of the expansions from the stable limit law, in the streamline of our approach.

We start to determine the correct scaling for the jumps of the random walk. First, observe that $\mathcal{E}^{-1}$ belongs to the domain of normal attraction of a stable law of index 1. Indeed, the tails distribution is

$$\mathbb{P}(\mathcal{E}^{-1} > x) = 1 - e^{-1/x} \sim x^{-1}, \qquad x \to +\infty.$$

Then, from e.g. Theorem 3.7.2 in [14],

$$\mathcal{S}^{(N)} := \frac{\sum_{i=1}^{N} \mathcal{E}_i^{-1} - b_N}{N} \xrightarrow{\text{law}} \mathcal{S}, \tag{3.26}$$

where $b_N = N\mathbb{E}(\mathcal{E}^{-1}; \mathcal{E}^{-1} < N)$, and $\mathcal{S}$ is the totally asymmetric stable law of index $\alpha = 1$, with characteristic function given for $u \in \mathbb{R}$ by

$$\begin{aligned} \mathbb{E}e^{iu\mathcal{S}} &= \exp\left\{ \int_1^{\infty} (e^{iux} - 1) \frac{dx}{x^2} + \int_0^1 (e^{iux} - 1 - iux) \frac{dx}{x^2} \right\} \\ &= \exp\left\{ iCu - \frac{\pi}{2} |u| \{ 1 + i \frac{2}{\pi} \mathrm{sign}(u) \ln |u| \} \right\} \\ &=: \exp \Psi_C(u), \end{aligned} \tag{3.27}$$

for some real constant C defined by the above equality. By integration by parts, one can check that, as $N \to \infty$,

$$b_N = N \int_{1/N}^{\infty} \frac{e^{-y}}{y} dy = N\big(\ln N - \gamma + \frac{1}{N} + \mathcal{O}(\frac{1}{N^2})\big), \tag{3.28}$$

with $\gamma = -\int_0^\infty e^{-x} \ln x dx$ the Euler constant. Then,

$$\ln b_N = \ln N + \ln\ln N - \frac{\gamma}{\ln N} + \mathcal{O}(\frac{1}{\ln^2 N})$$

We need to estimate

$$\begin{aligned} \mathbb{E} \ln \sum_{i=1}^N \mathcal{E}_i^{-1} - \ln b_N &= \mathbb{E} \ln \big(1 + \frac{N}{b_N} \mathcal{S}^{(N)}\big) \\ &= \mathbb{E} \ln \big(1 + \frac{N}{b_N} \mathcal{S}\big) + \mathcal{O}((\frac{1}{\ln N})^{1-\delta}), \end{aligned} \tag{3.29}$$

for all $\delta \in (0,1]$: indeed, since the moments of $\mathcal{S}^{(N)}$ of order $1-\delta/2$ are bounded (Lemma 5.2.2 in [17]), the sequence $(\frac{b_N}{N})^{1-\delta}\big[\ln\big(1+\frac{N}{b_N}\mathcal{S}^{(N)}\big) - \ln\big(1+\frac{N}{b_N}\mathcal{S}\big)\big]$ is uniformly integrable, and it converges to 0. A simple computation shows that

$$\mathbb{E}\ln(1+\varepsilon\mathcal{S}) = \int_1^\infty \ln(1+\varepsilon y)\frac{dy}{y^2}(1+o(1)) + \mathcal{O}(\varepsilon) \sim \varepsilon \ln(\varepsilon^{-1})$$

as $\varepsilon \searrow 0$. With $\varepsilon = N/b_N$, we recover the first 2 terms in the formula (3.24) for v_N. (If we could improve the error term in (3.29) to $o(\ln\ln N/\ln N)$, we would get also the third term.) With a similar computation, we estimate as $N \to \infty$

$$\begin{aligned} \sigma_N^2 &= \mathrm{Var}\Big(\ln\big(1+\frac{N}{b_N}\mathcal{S}^{(N)}\big)\Big) \\ &\sim \mathrm{Var}\Big(\ln\big(1+\frac{N}{b_N}\mathcal{S}\big)\Big) \\ &\sim \mathbb{E}\ln^2\big(1+\frac{N}{b_N}\mathcal{S}\big) \\ &\sim \int_1^\infty \ln^2(1+\frac{y}{\ln N})\frac{dy}{y^2} \\ &\sim \int_0^\infty \ln^2(1+\frac{y}{\ln N})\frac{dy}{y^2} \\ &= \frac{C_0}{\ln N}, \end{aligned}$$

with $C_0 = \int_0^\infty \ln^2(1+y)\frac{dy}{y^2} = \pi^2/3$.

3.3. **Scaling limit for large N.** In this section, we let the parameters a, λ of the Gumbel depend on N, and get stable law and process as scaling limits for the walk: In view of the above, we assume in this subsection that $\xi_{i,j} \sim \mathrm{G}(a,\lambda)$ where $a = a_N$ and $\lambda = \lambda_N$ depend on N,

$$\left\{ \begin{array}{lllll} \lambda_N &=& \frac{N}{b_N} & \sim & \frac{1}{\ln N}, \\ a_N &=& -C - \lambda_N^{-1}\ln(b_N) &=& -C - \ln^2 N - (\ln N)(\ln\ln N) + o(1), \end{array} \right. \tag{3.30}$$

with the constant C from (3.27). Correspondingly, we write

$$X = X^{(N)}, \Upsilon_N(t) = a_N + \lambda_N^{-1} \ln \left(\sum_{i=1}^{N} \mathcal{E}_i(t)^{-1} \right).$$

Note that, with $\mathcal{S}^{(N)}$ defined by the left-hand side of (3.26), we have by (3.21),

$$\begin{aligned} \Upsilon_N &= \frac{1}{\lambda_N} \ln \Big(\sum_{i=1}^{N} \mathcal{E}_i^{-1} \Big) - C - \frac{1}{\lambda_N} \ln b_N \\ &= \frac{1}{\lambda_N} \ln \Big(1 + \frac{N}{b_N} \mathcal{S}^{(N)} \Big) - C \\ &\xrightarrow{\text{law}} \mathcal{S}_0, \end{aligned} \tag{3.31}$$

as $N \to \infty$, where the stable variable $\mathcal{S}_0 = \mathcal{S} - C$ has characteristic function

$$\mathbb{E} \exp iu\mathcal{S}_0 = \exp \Psi_0(u), \qquad \Psi_0(u) = -\frac{\pi}{2}|u| - iu \ln |u|$$

from the particular choice of C. In words, with an appropriate renormalization as the system size increases, the instantaneous jump of the front converges to a stable law. For all integer n and independent copies $\mathcal{S}_{0,1}, \dots \mathcal{S}_{0,n}$ of $\mathcal{S}_0$, we see that

$$\frac{\mathcal{S}_{0,1} + \ldots + \mathcal{S}_{0,n}}{n} - \ln n \overset{\text{law}}{=} \mathcal{S}_0$$

from the characteristic function. Consider the totally asymmetric Cauchy process $(\mathcal{S}_0(\tau); \tau \geq 0)$, i.e. the independent increment process with characteristic function

$$\mathbb{E} \exp\{iu(\mathcal{S}_0(\tau) - \mathcal{S}_0(\tau'))\} = \exp\{(\tau - \tau')\Psi_0(u)\}, \qquad u \in \mathbb{R}, 0 < \tau' < \tau.$$

It is a Lévy process with Lévy measure x^{-2} on $\mathbb{R}_+$, it is not self-similar but it is stable in a wide sense: for all $\tau > 0$,

$$\frac{\mathcal{S}_0(\tau)}{\tau} - \ln \tau \overset{\text{law}}{=} \mathcal{S}_0(1)$$

with $\mathcal{S}_0(1) \overset{\text{law}}{=} \mathcal{S}_0$. We refer to [6] for a nice account on Lévy processes.

We may speed up the time of the front propagation as well, say by a factor $m_N \to \infty$ when $N \to \infty$, to get a continuous time description. Then, we consider another scaling, and define for $\tau > 0$,

$$\begin{aligned} \varphi_N(\tau) &= \frac{\Phi(X^{(N)}([m_N \tau])) - \Phi(X^{(N)}(0))}{m_N} - \tau \ln m_N \\ &= \frac{\sum_{t=1}^{[m_N \tau]} \Upsilon_N(t)}{m_N} - \tau \ln m_N \end{aligned} \tag{3.32}$$

by theorem 3.1. Of course, this new centering can be viewed as an additional shift in the formula (3.30) for a_N. By (3.31), the characteristic function $\chi_N(u) := \mathbb{E}e^{iu\Upsilon_N} = \exp\{\Psi_0(u)(1 + o(1))\}$, where $o(1)$ depends on u and tends to 0 as $N \to \infty$. Then,

$$\begin{aligned} \mathbb{E} \exp \left\{ iu(\frac{\sum_{t=1}^{[m_N \tau]} \Upsilon_N(t)}{m_N} - \tau \ln m_N) \right\} &= (\chi_N(u/m_N))^{[m_N \tau]} \exp\{-i\frac{u[m_N \tau]}{m_N} \ln m_N\} \\ &\to \exp \tau \Psi_0(u), \end{aligned}$$

as $N \to \infty$, showing convergence at a fixed time τ. In fact, convergence holds at the process level.

Theorem 3.2. *As $N \to \infty$, the process $\varphi_N(\cdot)$ converges in law in the Skorohod topology to the totally asymmetric Cauchy process $\mathcal{S}_0(\cdot)$.*

Proof. The process $\varphi_N(\cdot)$ itself has independent increments. The result follows from general results on triangular arrays of independent variables in the domain of attraction of a stable law, e.g. Theorems 2.1 and 3.2 in [18]. ∎

Proof of Theorem 1.1: Apply the previous Theorem 3.2 after making the substitution $\zeta = \lambda_N(\xi - a_N)$. ∎

4. The front profile as a traveling wave

Recall the front profile

$$U_N(t,x) = N^{-1} \sum_{i=1}^{N} \mathbf{1}_{X_i(t)>x} \tag{4.33}$$

which is a wave-like, random step function, traveling at speed v_N. One can write some kind of Kolmogorov-Petrovsky-Piscunov equation with noise (and discrete time) governing its evolution, see (7–10) in [7] and Proposition 4.1. Let F denote the distribution function of the ξ's, $F(x) = \mathbb{P}(\xi_{i,j}(t) \le x)$. Given $\mathcal{F}_{t-1}$, the right-hand side is, up to the factor N^{-1}, a binomial variable with parameters N and

$$\begin{aligned}
\mathbb{P}(X_i(t) > x|\mathcal{F}_{t-1}) &= 1 - \prod_{j=1}^{N} \mathbb{P}(X_j(t-1) + \xi_{i,j}(t) \le x|\mathcal{F}_{t-1}) \qquad \text{(by (1.1))} \\
&= 1 - \exp -N \int_{\mathbb{R}} \ln F(x-y) U_N(t-1, dy).
\end{aligned} \tag{4.34}$$

4.1. **Gumbel case.** Starting with the case of the Gumbel law $F(x) = \exp -e^{-\lambda(x-a)}$, we observe that (4.34) and (3.20) imply

$$\mathbb{P}(X_i(t) \le x|\mathcal{F}_{t-1}) = \exp -e^{\lambda(x-a-\Phi(X(t-1)))},$$

that is (3.23). It means that $X(t) - \Phi(X(t-1))$ is independent of $\mathcal{F}_{t-1}$, and that it is a N-sample of the law $G(a,\lambda)$. For the process at time t centered by the front location $\Phi(X(t-1))$, the product measure $G(a,\lambda)^{\otimes N}$ is invariant. We summarize these observations:

Proposition 4.1 ([7])**.** *Let $\xi_{i,j}(t) \sim \mathrm{G}(a,\lambda)$ be given, and X defined by (1.1). Then, the random variables $G_i(t)$ defined by $G(t) = (G_i(t); i \le N)$ and*

$$X(t) = G(t) + \Phi(X(t-1))\mathbf{1}, \quad t \ge 1, \tag{4.35}$$

are i.i.d. with common law $\mathrm{G}(a,\lambda)$, and $G(t)$ is independent of $X(t-1), X(t-2), \ldots$. In particular, $(G_i(t); i \le N, t \ge 1)$ is an i.i.d. sequence with law $G(a,\lambda)$, independent of $X(0) \in \mathbb{R}^N$. Moreover,

$$U_N(t,x) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}\{G_i(t) \ge x - \Phi(X(t-1))\}, \quad t \ge 1, x \in \mathbb{R}. \tag{4.36}$$

Remark 4.2. *(i) The recursion (4.36) is the* reaction-diffusion equation *satisfied by U_N. This equation is discrete and driven by a random noise* $(G(t); t \geq 0)$.

(ii) Note that the centering is given by a function of the configuration at the previous time $t-1$. One could easily get an invariant measure with a centering depending on the current configuration. For instance, consider

$$X(t) - \max_j X_j(t) \stackrel{\text{law}}{=} g - \max_j g_j,$$

with g_j i.i.d. $G(a, \lambda)$-distributed, or replace the maximum value by another order statistics. However our centering, allowing interesting properties like the representation (4.35), is the most natural.

By the law of large numbers, as $N \to \infty$, the centered front converges almost surely to a limit front, given by the (complement of) the distribution function of $\mathrm{G}(a, \lambda)$, as we state now.

Proposition 4.3. *For all $t \geq 1$, the following holds:*

(i) Convergence of the front profile: as $N \to \infty$, conditionally on $\mathcal{F}_{t-1}$, we have a.s.

$$U_N\big(t, x + \Phi(X(t-1))\big) \longrightarrow u(x) = 1 - \exp(-e^{-\lambda(x-a)}), \qquad \text{uniformly in } x \in \mathbb{R}.$$

(ii) Fluctuations: as $N \to \infty$,

$$\ln N \times \big\{U_N\big(t, x + (t-1)(\ln b_N + a) + \Phi(X(0))\big) - u(x)\big\} \stackrel{\text{law}}{\longrightarrow} \frac{u'(x)}{\lambda}\big(t\mathcal{S} + t\ln t + tC\big)$$

as $N \to \infty$, with $\mathcal{S}$ from (3.26) and C from (3.27).

We willl see in the proof that the front location alone is responsible for the fluctuations of the profile. It dominates a smaller Gaussian fluctuation due to the sampling.

Proof. of Proposition 4.3. As mentioned above, the law of large numbers yields pointwise convergence in the first claim. Since $U_N(t, \cdot)$ is non-inceasing, uniformity follows from Dini's theorem. We now prove the fluctuation result. By (3.21) and (3.26),

$$Z_N := \ln N \times \big\{\Phi(X(t)) - \Phi(X(0)) - t \ln b_N\big\} = \frac{\ln N}{\lambda} \sum_{s=1}^{t} \ln\big(1 + \frac{N}{b_N}\mathcal{S}^{(N)}(s)\big)$$

converges in law to the sum of t independent copies of $\mathcal{S}$, which has itself the law of $t\mathcal{S}+t\ln t+tC$. On the other hand, we have by (4.36),

$$U_N\big(t+1, x + t\ln b_N + \Phi(X(0))\big) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\{G_i(t+1) \geq x + \frac{1}{\ln N} Z_N\}.$$

By the central limit theorem for triangular arrays, for all sequences $z_N \to 0$, we see that,

$$N^{1/2}\big(\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\{G_i(t+1) \geq x + z_N\} - u(x+z_N)\big) \stackrel{\text{law}}{\longrightarrow} Z \sim \mathcal{N}(0, u(x)(1-u(x)))$$

as $N \to \infty$. Being of order $N^{-1/2}$, these fluctuations will vanish in front of the Cauchy ones, which are of order $(\ln N)^{-1}$. In the left hand side, we Taylor expand $u(x + z_N)$. Since $G(t+1)$ and Z_N are independent, we obtain

$$\ln N \times \big\{U_N\big(t+1, x + t\ln b_N + \Phi(X(0))\big) - u(x)\big\} - u'(x) Z_N \to 0$$

in probability, which proves the result. ∎

Remark: A limiting reaction-diffusion equation. It is natural to look for a reaction-diffusion equation which has u as traveling wave (soliton). By differentiation, one checks that, for all $v \in \mathbb{R}$, $\mathtt{u}(t,x) = u(x - vt)$ (where $u(x) = 1 - \exp\{-e^{-\lambda(x-a)}\}$) is a solution of

$$\mathtt{u}_t = \mathtt{u}_{xx} + A(\mathtt{u}), \tag{4.37}$$

with reaction term

$$A(u) = \lambda(1-u)\big[\lambda \ln \frac{1}{1-u} + (v - \lambda)\big] \ln \frac{1}{1-u}.$$

Since $A(0) = A(1) = 0$, the values $u = 0$ and $u = 1$ are equilibria. For $v \geq \lambda$, we have $A(u) > 0$ for all $u \in (0,1)$, hence these values are the unique equilibria $u \in [0,1]$, with $u = 0$ unstable and $u = 1$ stable. For $v \in [\lambda, 3\lambda)$, A is convex in the neighborhood of 0, so the equation is not of KPP type [16, p.2].

4.2. **Exponential tails: front profile and traveling wave.** In this section we prove Theorem 1.2. We consider the case of ξ with exponential upper tails, $1 - F(x) = \mathbb{P}(\xi > x) \sim e^{-x}$ as $x \to +\infty$, that can be written as

$$\lim_{x\to+\infty} \varepsilon(x) = 0, \qquad \text{with} \quad \varepsilon(x) = 1 + e^x \ln F(x). \tag{4.38}$$

(By affine transformation, we also cover the case of tails $\mathbb{P}(\xi > x) \sim e^{\lambda(x-a)}$.) By definition, $\varepsilon(x) \in [-\infty, 1]$.
We let $N \to \infty$, keeping t fixed and we use Φ from (3.20) with $\lambda = 1$. To show that the empirical distribution function (4.33) converges, after the proper shift, to that of the Gumbel distribution with the same tails, we will use the stronger assumption that

$$\lim_{x\to+\infty} \varepsilon(x) = 0, \qquad \text{and} \quad \varepsilon(x) \in [-\delta^{-1}, 1-\delta], \tag{4.39}$$

for all x with some $\delta > 0$.

Proof. (Theorem 1.2) First of all, note that $\ln F(x) = -(1-\varepsilon(x))e^{-x}$. Let $m_i = e^{X_i(t-1)-\Phi(X(t-1))}$, which add up to 1 by our choice of Φ, and let also $\varepsilon_i = \varepsilon(x + \Phi(X(t-1)) - X_i(t-1))$.
We start with the case $k = 1, K_N = \{j\}$. From (4.34),

$$\begin{aligned}
\ln \mathbb{P}(X_j(t) - \Phi(X(t-1)) &\leq x|\mathcal{F}_{t-1}) \\
&= \sum_{i=1}^{N} \ln F\big(x + \Phi(X(t-1)) - X_i(t-1)\big) \\
&= -\sum_{i=1}^{N} e^{-x-\Phi(X(t-1))+X_i(t-1)}[1 - \varepsilon(x + \Phi(X(t-1)) - X_i(t-1))] \\
&= -e^{-x}\sum_{i=1}^{N} m_i[1-\varepsilon_i] \\
&= -e^{-x}\left(1 - \sum_{i\in I_1} m_i\varepsilon_i - \sum_{i\in I_2} m_i\varepsilon_i\right)
\end{aligned} \tag{4.40}$$

with $I_1 = \{i : X_i(t-1) \le \Phi(X(t-1)) - A\}$ and I_2 the complement in $\{1, \dots N\}$, and some real number A to be chosen later. By the first assumption in (4.39), we have

$$|\sum_{i\in I_1} m_i\varepsilon_i| \le \sup\{|\varepsilon(y)|; y > x + A\} \times 1 \to 0 \quad \text{as } A \to \infty,$$

for fixed x. The second sum,

$$|\sum_{i\in I_2} m_i\varepsilon_i| \le \|\varepsilon\|_\infty \sum_{i\in I_2} e^{X_i(t-1)-\Phi(X(t-1))},$$

will be bounded using the second assumption in (4.39). We can enlarge the probability space and couple the $\xi_{i,j}(s)$'s with $(g_{i,j}(t-1); i, j \le N)$, which are i.i.d. $G(0,1)$ independent of $(\xi_{i,j}(s); i, j \le N, s \neq t-1)$, such that

$$g_{i,j}(t-1) + c \le \xi_{i,j}(t-1) \le g_{i,j}(t-1) + d.$$

Define for $i \le N$,

$$\tilde{X}_i(t-1) = \max_{j\le N}\{X_j(t-2) + g_{i,j}(t-1)\}.$$

By the previous double inequality,

$$\tilde{X}_i(t-1) + c \le X_i(t-1) \le \tilde{X}_i(t-1) + d,$$

and, since Φ is non-decreasing and such that $\Phi(y + r\mathbf{1}) = \Phi(y) + r$, we have also

$$\Phi(X(t-1)) - \Phi(X(t-2)) \ge \Phi(\tilde{X}(t-1)) - \Phi(X(t-2)) + c,$$

On the other hand, in analogy to the proof of Proposition 4.1 for the Gumbel case we know that $(\tilde{X}_i(t-1) - \Phi(X(t-2)); 1 \le i \le N)$ is a N-sample of the law $G(0,1)$. So,

$$\begin{aligned}\Phi(\tilde{X}(t-1)) - \Phi(X(t-2)) &= \ln(b_N) + \ln\left(1 + \frac{N}{b_N}\mathcal{S}^{(N)}\right)\\ &= \ln N + \ln\ln N + o(1)\end{aligned}$$

in probability from (3.26), and

$$\max\{\tilde{X}_i(t-1); i \le N\} - \Phi(X(t-2)) - \ln N \quad \text{converges in law}$$

by the limit law for the maximum of i.i.d.r.v.'s with exponential tails [20, Sect. I.6]. Combining these, we obtain, as $N \to \infty$,

$$\begin{aligned}\Phi(X(t-1)) - \max\{X_i(t-1); i \le N\} &\ge \Phi(\tilde{X}(t-1)) - d - \max\{\tilde{X}_i(t-1); i \le N\} + c\\ &\to +\infty \quad \text{in probability },\end{aligned}$$

which implies that the set I_2 becomes empty for fixed A and increasing N. This shows that $\sum_{i\in I_2} e^{X_i(t-1)-\Phi(X(t-1))} \to 0$ in probability (i.e., under $\mathbb{P}(\cdot|\mathcal{F}_{t-2})$) uniformly on $X(t-2)$. Letting $N \to \infty$ and $A \to +\infty$ in (4.40), we have

$$\mathbb{P}(X_j(t) - \Phi(X(t-1)) \le x|\mathcal{F}_{t-2}) \to \exp -e^{-x}$$

as $N \to \infty$ uniformly on $X(t-2)$, which implies the first claim for $k = 1$. For $k \ge 2$, recall that, conditionally on $\mathcal{F}_{t-1}$, the variables $(X_i(t); i \le N)$ are independent. The previous arguments apply, yielding (1.6).

Statement (1.7) for fixed x follows from this and the fact that $X_i(t)$ are independent conditionally on $\mathcal{F}_{t-1}$. Convergence uniform for x in compacts follows from pointwise convergence of monotone functions to a continuous limit (Dini's theorem). Uniform convergence on $\mathbb{R}$ comes from the additional property that these functions are bounded by 1. ∎

Remark 4.4. *(i) From the stochastic comparison (1.8) of ξ and the Gumbel, we obviously have $v_N = \ln b_N + \mathcal{O}(1)$. We believe, but could not prove, that the error term is in fact $o(1)$.*
(ii) We believe, but could not prove, that the conclusions of Theorem 1.2 hold under the only assumption that the function ε from (4.38) tends to 0 at $+\infty$.

5. Front speed for the Bernoulli distribution

In this section we consider the case of a Bernoulli distribution for the ξ's,

$$\mathbb{P}(\xi_{i,j}(t) = 1) = p, \qquad \mathbb{P}(\xi_{i,j}(t) = 0) = q = 1 - p,$$

with $p \in (0,1)$. For all starting configuration, from the coupling argument in the proof of proposition 2.1, we see that all N particles meet at a same location at a geometric time, and, at all later times, they share the location of the leading one, or they lye at a unit distance behind the leading one. We set $\Phi(x) = \max\{x_j; j \le N\}$, and we reduce the process X^0 to a simpler one given by considering

$$Z(t) = \sharp\big\{j : 1 \le j \le N, X_j(t) = 1 + \max\{X_i(t-1); i \le N\}\big\}. \tag{5.41}$$

$Z(t)$ is equal to the number of leaders if the front has moved one step forward at time t, and to 0 if the front stays at the same location. Here, we define the front location as the rightmost occupied site $\Phi(X(t)) = \max\{X_j(t); j \le N\}$. Then, it is easy to see that Z is a Markov chain on $\{0, 1, \ldots, N\}$ with transitions given by the binomial distributions

$$\mathbb{P}\big(Z(t+1) = \cdot\,|Z(t) = m\big) = \begin{cases} \mathcal{B}\big(N, 1 - q^m\big)(\cdot), & m \ge 1, \\ \mathcal{B}\big(N, 1 - q^N\big)(\cdot), & m = 0. \end{cases} \tag{5.42}$$

Note that the chain has the same law on the finite set $\{1, 2, \ldots\}$ when starting from 0 or from N. Clearly, $v_N \to 1$ as $N \to \infty$. We prove that the convergence is extremely fast.

Theorem 5.1. *In the Bernoulli case, we have*

$$v_N = 1 - q^{N^2} 2^N + o(q^{N^2} 2^N) \tag{5.43}$$

as $N \to \infty$.

Proof. The visits at 0 of the chain Z are the times when the front fails to move one step. Thus,

$$\Phi(X(t)) = \Phi(X(0)) + \sum_{s=1}^{t} \mathbf{1}_{Z(s) \ne 0},$$

which implies by dividing by t and letting $t \to \infty$, that

$$v_N = \bar{\nu}_N(Z \ne 0) = 1 - \bar{\nu}_N(Z = 0),$$

where $\bar{\nu}_N$ denotes the invariant (ergodic) distribution of the chain Z. Let E_N, P_N refer to the chain starting at N, and $T_k = \inf\{t \ge 1 : Z(t) = k\}$ the time of first visit at k $(0 \le k \le N)$. By Kac's lemma, we can express the invariant distribution, and get:

$$v_N = 1 - (E_0 T_0)^{-1} = 1 - (E_N T_0)^{-1}. \tag{5.44}$$

Let $\sigma_0 = 0$, and $\sigma_1, \sigma_2 \ldots$ the successive passage times of Z at N, and $\mathcal{N} = \sum_{i \ge 0} \mathbf{1}_{\sigma_i < T_0}$ the number of visits at N before hitting 0. Note that $\mathcal{N}$ has a geometric law with expectation

$E_N \mathcal{N} = P_N(T_0 < T_N)^{-1}$. Then,

$$
\begin{aligned}
E_N T_0 &= E_N \left[\sum_{i\geq 1} (\sigma_i - \sigma_{i-1}) \mathbf{1}_{\sigma_i < T_0} + (T_0 - \sigma_{\mathcal{N}}) \right] \\
&= \sum_{i\geq 1} E_N \left[(\sigma_i - \sigma_{i-1}) \mathbf{1}_{\sigma_i < T_0} \right] + E_N (T_0 - \sigma_{\mathcal{N}}) \\
&= \sum_{i\geq 1} E_N \left[\mathbf{1}_{\sigma_{i-1} < T_0} E_N \big(\sigma_1 \mathbf{1}_{\sigma_1 < T_0} \big) \right] + E_N (T_0 | T_0 < T_N) \qquad \text{(Markov property)} \\
&= E_N [\mathcal{N}] \times E_N \big(\sigma_1 \mathbf{1}_{\sigma_1 < T_0} \big) + E_N (T_0 | T_0 < T_N) \\
&= \frac{1 - P_N(T_0 < T_N)}{P_N(T_0 < T_N)} \times E_N \big(T_N | T_N < T_0 \big) + E_N (T_0 | T_0 < T_N) \qquad (5.45)
\end{aligned}
$$

We will prove a Lemma.

Lemma 5.1. *We have*

$$P_N(T_0 < T_N) \sim q^{N^2} 2^N, \qquad (5.46)$$

as N tends to ∞. Moreover,

$$\lim_{N\to\infty} E_N (T_0 | T_0 < T_N) = 2, \qquad (5.47)$$

$$\lim_{N\to\infty} E_N \big(T_N | T_N < T_0 \big) = 1. \qquad (5.48)$$

The lemma has a flavor of Markov chains with rare transitions considered in [9], except for the state space which is getting here larger and larger in the asymptotics. With the lemma at hand, we conclude that

$$
\begin{aligned}
E_N T_0 &\sim \frac{1}{P_N(T_0 < T_N)} \\
&\sim \frac{1}{q^{N^2} 2^N}
\end{aligned}
$$

as N tends to ∞. From (5.44), this implies the statement of the theorem. ∎

Proof. of lemma 5.1. We start to prove the key relation (5.46). We decompose the event $\{T_0 < T_N\}$ according to the number ℓ of steps to reach 0 from state N,

$$P_N(T_0 < T_N) = \sum_{\ell\geq 1} P_N(T_0 = \ell < T_N). \qquad (5.49)$$

We directly compute the contribution of $\ell = 1$: By (5.41), we have

$$P_N(T_0 = 1 < T_N) = q^{N^2}, \qquad (5.50)$$

which is neglegible in front the right-hand side of (5.46). We compute now the contribution of strategies in two steps:

$$\begin{aligned}
P_N(T_0 = 2 < T_N) &= \sum_{k=1}^{N-1} P_N(T_0 = 2 < T_N, Z(1) = k) \\
&= \sum_{k=1}^{N-1} \binom{N}{k} (1-q^N)^k q^{N(N-k)} \times \binom{N}{0} (1-q^k)^0 q^{kN} \\
&= q^{N^2} \sum_{k=1}^{N-1} \binom{N}{k} (1-q^N)^k \\
&= q^{N^2} \left[(2-q^N)^N - 1 - (1-q^N)^N \right] \\
&\sim q^{N^2} 2^N. \qquad (5.51)
\end{aligned}$$

For $\ell \geq 2$ we write, with the convention that $k_0 = N$,

$$\begin{aligned}
P_N(T_0 = \ell+1 < T_N) &= \sum_{k_1,\dots k_\ell=1}^{N-1} P_N(T_0 = \ell+1 < T_N, Z(i) = k_i, i = 1, \dots \ell) \\
&= \sum_{k_1,\dots k_\ell=1}^{N-1} \Big[\prod_{i=1}^{\ell} \binom{N}{k_i} (1-q^{k_{i-1}})^{k_i} q^{k_{i-1}(N-k_i)} \Big] q^{k_\ell N} \qquad \text{(by (5.41))} \\
&\leq \sum_{k_1,\dots k_\ell=1}^{N-1} \Big[\prod_{i=1}^{\ell} \binom{N}{k_i} q^{k_{i-1}(N-k_i)} \Big] q^{k_\ell N} \\
&= q^{N^2} \sum_{k_1,\dots k_\ell=1}^{N-1} \prod_{i=1}^{\ell} \binom{N}{k_i} q^{k_i(N-k_{i-1})} \qquad \text{(since } k_0 = N) \\
&=: q^{N^2} a_\ell, \qquad (5.52)
\end{aligned}$$

which serves as definition of $a_\ell = a_\ell(N)$. For $\varepsilon \in (0,1)$, define also $b_\ell = b_\ell(\varepsilon, N)$ by

$$b_\ell = \sum_{1 \leq k_1,\dots k_\ell \leq N-1, k_\ell > (1-\varepsilon)N} \prod_{i=1}^{\ell} \binom{N}{k_i} q^{k_i(N-k_{i-1})}$$

Then, by summing over k_ℓ,

$$\begin{aligned}
a_\ell &= \sum_{1 \leq k_1,\dots k_{\ell-1} \leq N-1} \Big[\prod_{i=1}^{\ell-1} \binom{N}{k_i} q^{k_i(N-k_{i-1})} \Big] \Big[(1+q^{N-k_{\ell-1}})^N - 1 - q^{N(N-k_{\ell-1})} \Big] \\
&\leq \sum_{1 \leq k_1,\dots k_{\ell-1} \leq N-1} \Big[\prod_{i=1}^{\ell-1} \binom{N}{k_i} q^{k_i(N-k_{i-1})} \Big] \Big[(1+q^{N-k_{\ell-1}})^N - 1 \Big] \\
&= \sum_{k_{\ell-1} \leq (1-\varepsilon)N} + \sum_{k_{\ell-1} > (1-\varepsilon)N} \\
&\leq \gamma_N \, a_{\ell-1} + (1+q)^N b_{\ell-1}, \qquad (5.53)
\end{aligned}$$

with

$$\gamma_N = \gamma_N(\varepsilon) \stackrel{\text{def.}}{=} \left(1 + q^{N\varepsilon}\right)^N - 1 \sim N q^{N\varepsilon}$$

as $N \to \infty$. We now bound b_ℓ in a similarly manner. First we note that, for any η such that $\eta > -\varepsilon \ln(\varepsilon) - (1-\varepsilon)\ln(1-\varepsilon) > 0$, we have

$$\sum_{k_\ell > (1-\varepsilon)N} \binom{N}{k_\ell} \le \exp N\eta \qquad \text{for large } N.$$

Note also that we can make η small by choosing ε small. Then,

$$\begin{aligned} b_\ell &\le \sum_{1 \le k_1, \ldots k_{\ell-1} \le N-1} \left[\prod_{i=1}^{\ell-1} \binom{N}{k_i} q^{k_i(N-k_{i-1})}\right] e^{N\eta} q^{(1-\varepsilon)N(N-k_{\ell-1})} \\ &= \sum_{k_{\ell-1} \le (1-\varepsilon)N} + \sum_{k_{\ell-1} > (1-\varepsilon)N} \\ &\le q^{\varepsilon(1-\varepsilon)N^2} e^{N\eta} a_{\ell-1} + q^{(1-\varepsilon)N} e^{N\eta} b_{\ell-1}. \end{aligned} \tag{5.54}$$

For vectors u, v, we write $u \le v$ if the inequality holds coordinatewise. In view of (5.53) and (5.54), we finally have

$$\begin{pmatrix} a_\ell \\ b_\ell \end{pmatrix} \le M \begin{pmatrix} a_{\ell-1} \\ b_{\ell-1} \end{pmatrix} \le \ldots \le M^{\ell-2} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \tag{5.55}$$

where the matrix M is positive and given by

$$M = \begin{pmatrix} \gamma_N & (1+q)^N \\ q^{\varepsilon(1-\varepsilon)N^2} e^{N\eta} & q^{(1-\varepsilon)N} e^{N\eta} \end{pmatrix}$$

We easily check that, for ε and η small, M has positive, real eigenvalues, and the largest one $\lambda_+ = \lambda_+(N, \varepsilon, \eta)$ is such that $\lambda_+ \sim \gamma_N$ as $N \to \infty$. By (5.55),

$$a_\ell \le \lambda_+^{\ell-2}(a_2 + b_2),$$

and since $\lambda_+ = \lambda_+(N, \varepsilon, \eta) < 1$ for large N,

$$\sum_{\ell \ge 2} a_\ell \le \frac{a_2 + b_2}{1 - \lambda_+}. \tag{5.56}$$

Now, we estimate $a_2 = a_2$ and b_2, both of which depend on N:

$$\begin{aligned} a_2 &= \sum_{1 \le k_1, k_2 \le N-1} \binom{N}{k_1} \binom{N}{k_2} q^{k_2(N-k_1)} \\ &\le \sum_{1 \le k_1 \le N-1} \binom{N}{k_1} \left[(1 + q^{N-k_1})^N - 1\right] \\ &= \sum_{k_1 \le (1-\varepsilon)N} + \sum_{k_1 > (1-\varepsilon)N} \\ &\le \gamma_N 2^N + (1+q)^N e^{N\eta}, \end{aligned}$$

and

$$b_2 \le q^{(1-\varepsilon)N} \sum_{1 \le k_1, k_2 \le N-1, k_2 > (1-\varepsilon)N} \binom{N}{k_1} \binom{N}{k_2} \le 2^N q^{(1-\varepsilon)N} e^{N\eta}.$$

From (5.56), we see that

$$\sum_{\ell \ge 2} a_\ell(N) = o(2^N),$$

and, together with (5.52), (5.49), (5.50), (5.51), it implies (5.46).

The limit (5.47) directly follows from the above estimates.

Finally, we turn to the proof of (5.48). Note that

$$P_N(T_N < T_0) \le E_N\big(T_N \mathbf{1}_{T_N<T_0}\big) \le E_N\big(T_N\big) = \Big(1-(1-p)^N\Big)^N + E_N\big(T_N \mathbf{1}_{T_N>1}\big) \,. \tag{5.57}$$

We only need to show that the last term is exponentially small. For that, we use Markov property at time 1,

$$E_N\big(T_N \mathbf{1}_{T_N>1}\big) \le \Big[1-\Big(1-(1-p)^N\Big)^N\Big]\Big(1+\max_m E_m(T_N)\Big),$$

where the first factor is exponentially small. To show that the second factor is bounded, one can repeat the proof of part a) with $p=1$ of the forthcoming Lemma 6.5. ∎

6. The case of variables taking a countable number of values

In this section we consider the case of a random variable ξ taking the values $\mathbb{N}_k := \{l \in \mathbb{Z} : l \le k\}$, with $k \in \mathbb{Z}$, so that

$$\mathbb{P}(\xi_{i,j}(t) = l) = p_l, \tag{6.58}$$

for $l \in \mathbb{N}_k$, with $p_l \ge 0, p_k \in (0,1)$ and $\sum_{l\in\mathbb{N}_k} p_l = 1$. As in the Bernoulli case we can reduce the process X^0 to a simpler one given by $Z(t) := (Z_l(t) : l \in \mathbb{N}_k)$, where

$$Z_l(t) = \sharp\big\{j : 1 \le j \le N, X_j(t) = \max\{X_i(t-1); 1 \le i \le N\} + l\big\}, \tag{6.59}$$

for $l \in \mathbb{N}_k$. Note that $Z_k(t)$ is equal to the number of leaders if the front has moved k steps forward at time t, and to 0 if the front moved less than k steps. Z is a Markov chain on the set

$$\Omega_k := \left\{ m \in \{0,\ldots,N\}^{\mathbb{N}_k} : \sum_{i\in\mathbb{N}_k} m_i = N \right\},$$

where m_i are the coordinates of m. We now proceed to compute the transition probabilities of the Markov chain Z. Assume that at some time t we have $Z_t = m = (m_i : i \in \mathbb{N}_k)$. For each $i \in \mathbb{N}_k$, this corresponds to m_i particles at position i. Let us now move each particle to the right adding independently a random variable with law $\xi_{0,0}$. We will assume that $m_k \ge 1$. The probability that at time $t+1$ there is some particle at position k is

$$s_k(m) := 1 - \left(\sum_{l=-\infty}^{k-1} p_l\right)^{m_k}.$$

Similarly, the probability that the rightmost particle at time $t+1$ is at position $k-1$ is

$$s_{k-1}(m) := \left(\sum_{l=-\infty}^{k-1} p_l\right)^{m_k} - \left(\sum_{l=-\infty}^{k-1} p_l\right)^{m_{k-1}} \left(\sum_{l=-\infty}^{k-2} p_l\right)^{m_k}.$$

In general, for $r \in \mathbb{N}_k$, the probability that at time $t+1$ the rightmost particle is at position r is

$$\begin{aligned} s_r(m) \;&:=\; \left(\sum_{l=-\infty}^{k-1} p_l\right)^{m_{r+1}} \cdots \left(\sum_{l=-\infty}^{r} p_l\right)^{m_k} \\ &\qquad - \left(\sum_{l=-\infty}^{k-1} p_l\right)^{m_r} \cdots \left(\sum_{l=-\infty}^{r} p_l\right)^{m_{k-1}} \left(\sum_{l=-\infty}^{r-1} p_l\right)^{m_k}. \end{aligned}$$

Define now on Ω_k the shift θm by $(\theta m)_i = m_{i-1}$ for $i \in \mathbb{N}_k$. For $r \in \mathbb{N}_k$, let $s_r^{(1)}(m) := s_r(\theta m)$ and in general for $j \geq 1$ let

$$s_r^{(j)}(m) := s_r(\theta^j m).$$

Define $s(m) := (s_r(m) : r \in \mathbb{N}_k)$, for $j \geq 1$, $s^{(j)}(m) := (s_r^{(j)}(m) : r \in \mathbb{N}_k)$. Dropping the dependence on m of s_r, $s_r^{(j)}$, s and $s^{(j)}$ we can now write the transition probabilities of the process $Z(t)$ as

$$\mathbb{P}\big(Z(t+1) = n | Z(t) = m\big) = \begin{cases} \mathcal{M}\big(N; s\big)(n), & m_k \geq 1, \\ \mathcal{M}\big(N; s^{(1)}\big)(n), & m_{k-1} \geq 1, m_k = 0, \\ \mathcal{M}\big(N; s^{(2)}\big)(n), & m_{k-2} \geq 1, m_k = m_{k-1} = 0, \\ \dots & \dots \\ \mathcal{M}\big(N; s^{(j)}\big)(n), & m_{k-j} \geq 1, m_k = m_{k-1} = \cdots = m_{k-j+1} = 0, \end{cases} \tag{6.60}$$

where for u_i with $\sum_i u_i = 1$, $\mathcal{M}\big(N; u\big)$ denotes the multinomial distribution (with infinitely many classes). Let us introduce the following notation

$$r_i := \sum_{j=-\infty}^{k-i} p_j,$$

for integer $i \geq 1$.

Assumption (R). We say that a random variable ξ distributed according to (6.58) satisfies assumption (R) if

$$p_k \times p_{k-1} > 0$$

and

$$\mathbb{E}(|\xi_{0,0}|) < \infty.$$

We can now state the main result of this section.

Theorem 6.1. *Let ξ be distributed according to (6.58) and suppose that it satisfies assumption (R). Then, we have that*

$$v_N = k - q_k^{N^2} 2^N + o(q_k^{N^2} 2^N), \tag{6.61}$$

as $N \to \infty$, where $q_k := 1 - p_k$.

6.1. **Proof of Theorem 6.1.** To prove Theorem 6.1, we will follow a strategy similar to the one used in the Bernoulli case. Let us first define for each $m = (m_i : i \in \mathbb{N}_k) \in \Omega_k$ the function

$$\phi = \phi(m) := \sup\{i \in \mathbb{N}_k : m_i > 0\}. \tag{6.62}$$

As in the Bernoulli case, we denote by $\bar{\nu}_N$ the invariant (ergodic) distribution of the chain Z.

Lemma 6.1. *Let ξ be distributed according to (6.58). Then, we have that*

$$v_N = k - \bar{\nu}_N(\phi \leq k-1) - \sum_{j=2}^{\infty} \bar{\nu}_N(\phi \leq k-j). \tag{6.63}$$

Proof. Let $\Phi(x) = \max\{x_i; i \le N\}$, and note that for every positive integer time t

$$\Phi(X(t)) = \Phi(X(0)) + \sum_{i=1}^{t} \phi(Z(i)).$$

Hence

$$v_N = \sum_{i \in \mathbb{N}_k} i\bar{\nu}_N(\phi = i) = k - \sum_{j=1}^{\infty} \bar{\nu}_N(\phi \le k - j). \tag{6.64}$$

∎

We will now show that the first two terms of the expression for the velocity (6.63) given in Lemma 6.1, dominate the others.

Lemma 6.2. *Let ξ be distributed according to (6.58). Then, for each $i \ge 2$ we have that*

$$\bar{\nu}_N(\phi \le k - i) \le \left(\frac{r_i}{r_1}\right)^N \bar{\nu}_N(\phi \le k - 1).$$

Proof. Let us fix $m \in \Omega_k$. Define $\kappa := \sup\{i \in \mathbb{N}_k : m_i > 0\}$. Let us first note that

$$P_m(\phi(Z(1)) \le k - 1) = r_1^{m_\kappa N},$$

while

$$P_m(\phi(Z(1)) \le k - 2) = r_1^{m_{\kappa-1} N} r_2^{m_\kappa N} \le \left(\frac{r_2}{r_1}\right)^{m_\kappa N} P_m(\phi(Z(1)) \le k - 1).$$

Hence

$$\begin{aligned}
P_m(\phi(Z(t)) \le k - 2) &= \sum_{m' \in \Omega_k} P_m(Z(t-1) = m', \phi(Z(t)) \le k - 2) \\
&= \sum_{m' \in \Omega_k} P_m(Z(t-1) = m') P_{m'}(\phi(Z(1)) \le k - 2) \\
&\le \sum_{m' \in \Omega_k} P_m(Z(t-1) = m') P_{m'}(\phi(Z(1)) \le k - 1) \left(\frac{r_2}{r_1}\right)^{m'_\kappa N} \\
&\le \left(\frac{r_2}{r_1}\right)^N P_m(\phi(Z(t)) \le k - 1),
\end{aligned}$$

where in the last inequality we used the fact that by definition $m'_{\kappa'} \ge 1$. A similar reasoning shows that in general, for $i \ge 2$,

$$P_m(\phi(Z(t)) \le k - i) \le \left(\frac{r_i}{r_1}\right)^N P_m(\phi(Z(t)) \le k - 1).$$

Taking the limit when $t \to \infty$ and using Proposition 2.1, we conclude the proof. ∎

Lemma 6.3. *Let ξ be distributed according to (6.58) and suppose that assumption (R) is satisfied. Then*

$$\sum_{i=2}^{\infty}\left(\frac{r_i}{r_1}\right)^N = \mathcal{O}\left(\left(\frac{r_2}{r_1}\right)^N\right)$$

Proof. Note that by summation by parts, assumption (R) implies that

$$\sum_{i=2}^{\infty} r_i < \infty.$$

Therefore,

$$\sum_{i=2}^{\infty}\left(\frac{r_i}{r_1}\right)^N \leq \frac{1}{r_1}\left(\frac{r_2}{r_1}\right)^{N-1}\sum_{i=2}^{\infty} r_i = \mathcal{O}\left(\left(\frac{r_2}{r_1}\right)^N\right).$$

∎

Theorem 6.1 now follows from Lemmas 6.1, 6.2, 6.3 and the next proposition, whose proof we defer to subsection 6.2.

Proposition 6.2. *We have that*

$$\lim_{N\to\infty}\frac{\bar{\nu}_N(\phi \leq k-1)}{q_k^{N^2}2^N} = 1.$$

6.2. **Proof of Proposition 6.2.** Let us introduce for each $m \in \Omega_k$ the stopping time

$$T_m := \inf\{t \geq 1 : Z(t) = m\}.$$

Define now $\Omega_k^0 := \{m \in \Omega_k : m_k = 0\}$. Furthermore, we denote in this section $\oplus := (\dots, 0, N) \in \Omega_k$. We now note that by Kac's formula

$$\bar{\nu}_N(Z_k = 0) = \sum_{n\in\Omega_k^0}\bar{\nu}_N(Z = n) = \sum_{n\in\Omega_k^0}\frac{1}{E_n(T_n)}.$$

Hence we have to show that

$$\lim_{N\to\infty}\frac{\sum_{n\in\Omega_k^0}\frac{1}{E_n(T_n)}}{q_k^{N^2}2^N} = 1. \tag{6.65}$$

We will prove (6.65) through the following three lemmas.

Lemma 6.4. *Assume that ξ is distributed according to (6.58). Then, for every $n \in \Omega_k^0$ we have that*

$$E_n(T_n) = E_\oplus(T_\oplus, T_\oplus < T_n)\frac{1}{P_\oplus(T_n < T_\oplus)} + E_\oplus(T_n|T_n < T_\oplus) + U_N(n),$$

where $1 - e^{-CN} \leq \inf_{n_1,\dots,n_{k-1},0}|U_N| \leq \sup_{n_1,\dots,n_{k-1},0}|U_N| \leq 2 + e^{-CN}$ for some constant $C > 0$.

Lemma 6.5. *Assume that ξ is distributed according to (6.58). Then, there is a constant $C > 0$ such that the following are satisfied.*

a) *For $p = 1$ and $p = 2$, and for every $N \geq 2$ we have that*

$$\sup_{m\in\Omega_k} E_m(T_\oplus^p) \leq 2^p(1 + e^{-CN}). \tag{6.66}$$

b) *For every $N \geq 2$ we have that*

$$\sup_{m\in\Omega_k^0} |E_\oplus(T_\oplus, T_\oplus < T_m) - 1| \leq e^{-CN}.$$

To state the third lemma, we need to define the first hitting time of the set Ω_k^0. We let

$$T_A := \inf_{m\in\Omega_k^0} T_m.$$

Lemma 6.6. *Assume that ξ is distributed according to (6.58). Then, there is a constant $C > 0$ such that*

$$\sum_{n\in\Omega_k^0} P_\oplus(T_n < T_\oplus) = P_\oplus(T_A < T_\oplus)\big(1 + \mathcal{O}(e^{-CN})\big).$$

Let us now see how Lemmas 6.4, 6.5 and 6.6 imply Proposition 6.2. We will see that in fact, Proposition 6.2 will follow as a corollary of the corresponding result for the Bernoulli case with $q = q_k$. Note that Lemma 6.4 and part (b) of Lemma 6.5 imply that

$$P_\oplus(T_n < T_\oplus) \geq \frac{1 - e^{-CN}}{E_n(T_n)}, \qquad n \in \Omega_k^0.$$

Hence, summing up over $n \in \Omega_k^0$, by Lemma 6.6, we get that, for some $C' > 0$,

$$P_\oplus(T_A < T_\oplus) \geq (1 - e^{-C'N}) \sum_{n\in\Omega_k^0} \frac{1}{E_n(T_n)}. \tag{6.67}$$

Now, note that $P_\oplus(T_A < T_\oplus)$ is equal to the probability to hit 0 before N, starting from N, for the chain Z defined through random variables with Bernoulli increments as in Section 5. Hence, by (5.46) of Lemma 5.1 we conclude that for N large enough

$$(1 + e^{-CN})q_k^{N^2} 2^N \geq \sum_{n\in\Omega_k^0} \frac{1}{E_n(T_n)}. \tag{6.68}$$

On the other hand, applying the Cauchy-Schwarz inequality to the expectation $E_\oplus(\cdot|T_n < T_\oplus)$ in Lemma 6.4 and using Lemma 6.5, we obtain for each $n \in \Omega_k^0$ that

$$E \leq \frac{a_1}{P} + \frac{a_2}{\sqrt{P}} + a_3,$$

where $a_1 := 1 + e^{-CN}$, $a_2 := 2(1 + e^{-CN})$ and $a_3 := U_N$, $E := E_n(T_n)$, $P := P_\oplus(T_n < T_\oplus)$ and we have used (6.66) of part (a) of Lemma 6.5 with $p = 2$. It follows that

$$\frac{1}{\sqrt{P}} \geq \frac{\sqrt{a_2^2 - 4a_1(a_3 - E)} - a_2}{2a_1}.$$

Hence,

$$a_1 \frac{1}{P} \geq E - \frac{a_2}{2a_1}\sqrt{a_2^2 - 4a_1(a_3 - E)}.$$

Now, $a_2^2 - 4a_1(a_3 - E) \le 8(1+E)$ for large N, so that

$$a_1 \frac{1}{P} \ge E\left(1 - 4\frac{1}{\sqrt{E}}\sqrt{8\left(\frac{1}{E}+1\right)}\right).$$

Now, by inequality (6.68) we conclude that for N large enough $\frac{1}{E} \le q_k^{N^2} 2^{N+1}$. Therefore,

$$\frac{1}{E_n(T_n)} \ge (1 - e^{-C'N}) P_\oplus(T_n < T_\oplus).$$

Summing up over $n \in \Omega_k^0$, by Lemma 6.6 we get that

$$P_\oplus(T_A < T_\oplus) \le (1 + e^{-C'N}) \sum_{n \in \Omega_k^0} \frac{1}{E_n(T_n)} \tag{6.69}$$

for some $C' > 0$. Finally, (5.46) of Lemma 5.1, together with inequalities (6.67) and (6.69), imply inequality (6.65), which finishes the proof of Proposition 6.2. ∎

6.2.1. *Proof of Lemma 6.5. Part (a).* We will first prove that there exists a constant $C > 0$ such that

$$\sup_{m \in \Omega_k} P_m(T_\oplus > 2) \le e^{-CN}. \tag{6.70}$$

The strategy to prove this bound will be to show that with a high probability, after one step there are at least $\frac{p_k N}{2}$ leaders. This gives a high probability of then having N leaders in the second step. Consider now the set $L_{k,N} := \left\{ m \in \Omega_k : m_k \ge \left[\frac{p_k N}{2}\right] \right\}$. We have

$$P_m(T_\oplus \le 2) \ge P\left(X \ge \frac{p_k N}{2}\right) \inf_{m \in L_{k,N}} P_m(T_\oplus = 1), \tag{6.71}$$

where X is a random variable with a binomial distribution of parameters p_k and N. Now, by a large deviation estimate, the first factor of (6.71) is bounded from below by $1 - e^{-CN}$. On the other hand, we have for $m \in L_{k,N}$,

$$P_m(T_\oplus = 1) \ge \left(1 - (1-p_k)^{Np_k/2}\right)^N \ge 1 - e^{-CN},$$

for some constant $C > 0$. This estimate combined with (6.71) proves inequality (6.70). Now, by the Markov property, we get that, for all $m \in \Omega_k$,

$$\begin{aligned}
E_m(T_\oplus) &= E_m(T_\oplus 1_{T_\oplus \le 2}) + \sum_{n \in \Omega_k} E_m(T_\oplus 1_{T_\oplus > 2, Z(2) = n}) \\
&\le 2P_m(T_\oplus \le 2) + \sum_{n \in \Omega_k} E_m\left(1_{T_\oplus > 2, Z(2) = n}[2 + E_n(T_\oplus)]\right) \\
&\le 2P_m(T_\oplus \le 2) + \left(2 + \sup_{n \in \Omega_k} E_n(T_\oplus)\right) P_m(T_\oplus > 2),
\end{aligned}$$

where the supremum is finite, in fact smaller than δ_N^{-1} with δ_N from (2.15). Bounding the first term of the right-hand side of the above inequality by 2, taking the supremum over $m \in \Omega_k$ and applying the bound (6.70), we obtain (6.66) of (a) of Lemma 6.5 with $p = 1$. The proof of (6.66) when $p = 2$ is analogous via an application of the case $p = 1$.

Part (b). Note that for every state $m \in \Omega_k^0$ we have that

$$E_\oplus(T_\oplus, T_\oplus < T_A) \le E_\oplus(T_\oplus, T_\oplus < T_m) \le E_\oplus(T_\oplus).$$

Hence, it is enough to prove that

$$|E_\oplus(T_\oplus) - 1| \le e^{-CN}, \tag{6.72}$$

and that

$$|E_\oplus(T_\oplus, T_\oplus < T_A) - 1| \le e^{-CN}. \tag{6.73}$$

To prove (6.72) note that

$$E_\oplus(T_\oplus) = \left(1 - (1-p_k)^N\right)^N + E_\oplus(T_\oplus, T_\oplus > 1). \tag{6.74}$$

But by the Markov property,

$$E_\oplus(T_\oplus, T_\oplus > 1) \le \left(1 - \left(1 - (1-p_k)^N\right)^N\right)\left(1 + \sup_{m\in\Omega_k^0} E_m(T_\oplus)\right).$$

Note that

$$\left(1 - (1-p_k)^N\right)^N \ge \exp\left\{-\frac{N(1-p_k)^N}{1-N(1-p_k)^N}\right\} \ge 1 - \frac{N(1-p_k)^N}{1-N(1-p_k)^N}.$$

Using part (a) just proven of this Lemma, we conclude that

$$E_\oplus(T_\oplus, T_\oplus > 1) \le e^{-CN}. \tag{6.75}$$

Substituting this back into (6.74) we obtain inequality (6.72). To prove inequality (6.73), as before, observe that

$$E_\oplus(T_\oplus, T_\oplus < T_A) = \left(1 - (1-p_k)^N\right)^N + E_\oplus(T_\oplus, T_A > T_\oplus > 1). \tag{6.76}$$

Noting that $E_\oplus(T_\oplus, T_A > T_\oplus > 1) \le E_\oplus(T_\oplus, T_\oplus > 1)$, we can use the estimate (6.75) to obtain (6.73).

6.2.2. *Proof of Lemma 6.4.* We will use the following relation, which proof is similar to that of (5.45) and will be not be repeated here: for every $n \in \Omega_k^0$,

$$E_\oplus(T_n) = E_\oplus(T_\oplus | T_\oplus < T_n)\frac{P_\oplus(T_\oplus < T_n)}{P_\oplus(T_n < T_\oplus)} + E_\oplus(T_n | T_n < T_\oplus). \tag{6.77}$$

Let us now derive Lemma 6.4. Let $n \in \Omega_k^0$ and $m \in \Omega_k$. We first make the decomposition

$$E_m(T_n) = (T)_1 + (T)_2, \tag{6.78}$$

where

$$(T)_1 := E_m(T_n 1_{T_\oplus < T_n}) \qquad \text{and}$$
$$(T)_2 := E_m(T_n 1_{T_\oplus > T_n}).$$

We also denote by $\overline{(T)_2}$ the supremum of $(T)_2$ over all possible $n \in \Omega_k^0$ and $m \in \Omega_k$. Now,

$$(T)_2 = (T)_{21} + (T)_{22}, \tag{6.79}$$

where

$$(T)_{21} := E_m(T_n 1_{T_\oplus > T_n} 1_{Z_k(1) > CN}) \qquad \text{and}$$
$$(T)_{22} := E_m(T_n 1_{T_\oplus > T_n} 1_{Z_k(1) \leq CN}).$$

Now note that for any constant $C < p_k$, by the Markov property and a standard large deviation estimate we have that

$$\begin{aligned} (T)_{22} &= P_m(T_n = 1) + \sum_{z_1 \leq CN} E_m(T_n 1_{T_\oplus > T_n \geq 2} 1_{Z_k(1) = z_1}) \\ &\leq P_m(Z(1) = n) + \left(1 + \overline{(T)_2}\right) P_m(Z_k(1) \leq CN, Z(1) \neq n) \\ &\leq \left(1 + \overline{(T)_2}\right) P_m(Z_k(1) \leq CN) \\ &\leq \left(1 + \overline{(T)_2}\right) e^{-cN}, \end{aligned} \tag{6.80}$$

for some constant $c > 0$ depending on C, p_k. On the other hand, by definition of the event $\{T_\oplus > T_n\}$, we have the first equality below:

$$\begin{aligned} (T)_{21} &= E_m(T_n 1_{T_\oplus > T_n} 1_{Z_k(1) > CN} 1_{Z_k(2) \leq N-1}) \\ &\leq \left(1 - (1 - (1 - p_k)^{CN})^N\right) (2 + \overline{(T)_2}) \\ &\leq C' N (1 - p_k)^{CN} (2 + \overline{(T)_2}), \end{aligned} \tag{6.81}$$

for some $C' > 0$. We can now conclude from (6.79), (6.80) and (6.81), that there is a constant $C > 0$ such that

$$\overline{(T)_2} \leq C e^{-CN}.$$

Let us now take $m = n \in \Omega_k^0$ and examine the first term of the decomposition (6.78). Note that by the strong Markov property,

$$(T)_1 = E_\oplus(T_n) + E_n(T_\oplus 1_{T_\oplus < T_n}). \tag{6.82}$$

Now, by part (a) Lemma 6.5 with $p = 1$, we see that the second term in the above decomposition is bounded above as follows,

$$E_n(T_\oplus) \leq 2(1 + e^{-CN}). \tag{6.83}$$

Collecting our estimates, we get

$$\begin{aligned} E_n T_n &= E_n(T_n; T_\oplus < T_n) + E_n(T_n; T_n < T_\oplus) \\ &= E_n(T_\oplus; T_\oplus < T_n) + E_n(T_n - T_\oplus; T_\oplus < T_n) + E_n(T_n; T_n < T_\oplus) \\ &= E_n(T_\oplus; T_\oplus < T_n) + P_n(T_\oplus < T_n) \times E_\oplus(T_n) + E_n(T_n; T_n < T_\oplus). \end{aligned}$$

Here we bound the first term with (6.83), the last one by $\overline{(T)_2}$, and we can use (6.77) to obtain the desired conclusion. ∎

6.2.3. *Proof of Lemma 6.6.* First note that

$$\sum_{n \in \Omega_k^0} P_\oplus(T_n < T_\oplus) \geq P_\oplus(T_A < T_\oplus),$$

and it suffices to prove an inequality in the converse direction. It is natural to introduce the number $\mathcal{N}_A$ of visits of the chain to the set Ω_k^0 before reaching the $\oplus$ state,

$$\mathcal{N}_A := \sum_{t=1}^{T_\oplus} \mathbf{1}_{Z(t)\in\Omega_k^0} \ ,$$

since we have, for all $m \in \Omega_k$, the relations

$$E_m\mathcal{N}_A \geq \sum_{n\in\Omega_k^0} P_m(T_n < T_\oplus) \ , \quad P_m(\mathcal{N}_A \geq 1) = P_m(T_A < T_\oplus) \ . \tag{6.84}$$

Then, by the strong Markov property,

$$\begin{aligned} E_\oplus(\mathcal{N}_A) &= E_\oplus(\mathcal{N}_A 1_{\mathcal{N}_A\geq 1}) \\ &= \sum_{n\in\Omega_k^0} E_\oplus\left(1_{T_A<T_\oplus, Z(T_A)=n} E_n(1+\mathcal{N}_A)\right) \\ &\leq \left(1 + \sup_{n\in\Omega_k^0} E_n(\mathcal{N}_A)\right) P_\oplus(\mathcal{N}_A \geq 1) \ . \end{aligned} \tag{6.85}$$

In view of (6.84), where the first term is smaller than the last one, it suffices to show that

$$\sup_{n\in\Omega_k^0} E_n(\mathcal{N}_A) = \mathcal{O}(e^{-CN})$$

in order to conclude the proof of the Lemma. In this purpose, use the strong Markov property to write

$$\begin{aligned} E_n(\mathcal{N}_A) &= E_n\left(\mathcal{N}_A 1_{T_\oplus=1}\right) + E_n\left(\mathcal{N}_A 1_{T_\oplus\geq 2}\right) \\ &= 0 + \sum_{m\in\Omega_k^0} E_n\left(1_{T_A<T_\oplus, Z(T_A)=m}(1+E_m\mathcal{N}_A)\right) \\ &\leq \left(1 + \sup_{m\in\Omega_k} E_m(\mathcal{N}_A)\right) P_n(T_A < T_\oplus) \ . \end{aligned} \tag{6.86}$$

Observe also that, for all $n \in \Omega_k$,

$$\begin{aligned} P_n(T_A < T_\oplus) &\leq P_n(T_A = 1) + P_n(T_\oplus > 2) \\ &\leq (1-p_k)^N + \sup_{n\in\Omega_k} P_n(T_\oplus > 2) \\ &\leq 2e^{-CN} \end{aligned} \tag{6.87}$$

by (6.70). Now, the desired result follows from (6.86) and (6.87), provided that the supremum in the former estimate is finite. To show this, note that $\sup_m P_m(T_\oplus \geq 2) \leq (1-p_k)^N$, which implies that $T_\oplus$ is stochastically dominated by a geometric variable with this parameter. Therefore,

$$\sup_m E_m(\mathcal{N}_A) \leq \sup_m E_m(T_\oplus) \leq (1-p_k)^{-N},$$

ending the proof. ∎

6.3. **Proof of Theorem 1.3.** Changing ξ into $(\xi - a)/(a - b)$, we can restrict to the case $a = 0, b = -1$. Then, for fixed $\varepsilon > 0$, we define i.i.d. sequences $\hat{\xi}_{i,j}(t)$ and $\breve{\xi}_{i,j}(t)$ by

$$\hat{\xi}_{i,j}(t) = -\mathbf{1}_{\{\xi_{i,j}(t) \le -1\}} \ , \qquad \breve{\xi}_{i,j}(t) = (1+\varepsilon) \sum_{\ell \le -1} \ell \mathbf{1}_{\{\xi_{i,j}(t) \in [\ell(1+\varepsilon), (\ell+1)(1+\varepsilon))\}} \ .$$

Clearly, these variables are integrable since ξ is. Since $\breve{\xi}_{i,j}(t) \le \xi_{i,j}(t) \le \hat{\xi}_{i,j}(t)$, the corresponding speeds are such that

$$\breve{v}_N \le v_N \le \hat{v}_N \ .$$

From Theorem 6.1, both $\hat{v}_N$ and $(1+\varepsilon)^{-1}\breve{v}_N$ are $-(1-p)^{N^2}2^N + o((1-p)^{N^2}2^N)$ as $N \to \infty$, which, in addition to the previous inequalities, yields

$$-(1+\varepsilon) \le \liminf_{N\to\infty} v_N(1-p)^{-N^2}2^{-N} \le \limsup_{N\to\infty} v_N(1-p)^{-N^2}2^{-N} \le -1.$$

Letting $\varepsilon \searrow 0$, we obtain the desired claim. ∎

References

[1] M. Balázs, M. Z. Rácz, B. Tóth: Modeling flocks and prices: jumping particles with an attractive interaction. `http://arxiv.org/abs/1107.3289`, to appear in *Ann. Inst. H. Poincaré – Probabilités et Statistiques*

[2] J. Baik, P. Deift, Percy, K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* **12** (1999), 1119–1178.

[3] J. Berestycki, N. Berestycki, J. Schweinsberg: Survival of near-critical branching Brownian motion. *J. Stat. Phys.* **143** (2011), 833–854

[4] J. Bérard, J.-B. Gouéré: Brunet-Derrida behavior of branching-selection particle systems on the line. *Comm. Math. Phys.* **298** (2010), 323–342.

[5] J. Bérard, J.-B. Gouéré: Survival probability of the branching random walk killed below a linear boundary. *Electron. J. Probab.* **16** (2011), 396–418.

[6] J. Bertoin: *Lévy processes.* Cambridge Tracts in Math. 121. Cambridge Univ. Press, Cambridge, 1996

[7] E. Brunet and B. Derrida: Exactly soluble noisy traveling-wave equation appearing in the problem of directed polymers in a random medium. *Phys. Rev. E* **70** (2004), 016106

[8] E. Brunet, B. Derrida, A. Mueller, S. Munier: Noisy traveling waves: effect of selection on genealogies. *Europhys. Lett.* **76** (2006), no. 1, 1–7.

[9] O. Catoni, R. Cerf: The exit path of a Markov chain with rare transitions. *ESAIM Probab. Statist.* **1** (1995/97), 95–144

[10] S. Chatterjee, S. Pal: A phase transition behavior for Brownian motions interacting through their ranks. *Probab. Theory Related Fields* **147** (2010) 123–159

[11] O. Couronné, L. Gérin: Survival Time of a Censored Supercritical Galton-Watson Process. `http://fr.arxiv.org/abs/1111.1078` , to appear in *Ann. Inst. H. Poincaré – Probabilités et Statistiques*

[12] A. Dembo, O. Zeitouni: *Large deviations techniques and applications.* Applications of Mathematics 38. Springer-Verlag, New York, 1998

[13] B. Derrida, H. Spohn : Polymers on disordered trees, spin glasses, and traveling waves. *J. Statist. Phys.*, **51** (1988), 817–840.

[14] R. Durrett: *Probability: theory and examples.* Fourth edition. Cambridge University Press, Cambridge, 2010.

[15] R. Durrett, D. Remenik: Brunet-Derrida particle systems, free boundary problems and Wiener-Hopf equations. To appear in *Annals of Probability*, 2011

[16] B. Gilding, R. Kersner: *Travelling waves in nonlinear diffusion-convection reaction.* Progress in Nonlinear Differential Equations and their Applications, 60. Birkhäuser Verlag, Basel, 2004

[17] I. Ibragimov, Y. Linnik: *Independent and stationary sequences of random variables.* Wolters-Noordhoff Publishing, Groningen, 1971.

[18] J. Jacod. Théorèmes limite pour les processus. Saint-Flour notes 1983, 298–409, Lecture Notes in Math., 1117, Springer.

[19] K. Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.* **209** (2000), 437–476

[20] M. Leadbetter, G. Lindgren, H. Rootzén: *Extremes and related properties of random sequences and processes*. Springer Series in Statistics. Springer-Verlag, New York-Berlin, 1983

[21] P. Maillard: Branching Brownian motion with selection of the N right-most particles: An approximate model. `http://arxiv.org/abs/1112.0266`, to appear in *Ann. Inst. H. Poincaré – Probabilités et Statistiques*

[22] S.N. Majumdar, P. Krapivsky: Extreme value statistics and traveling fronts: Application to computer science *Phys. Rev. E* **65** (2002), 036127

[23] C. Monthus, T. Garel: Anderson transition on the Cayley tree as a traveling wave critical point for various probability distributions. *J. Phys. A* **42** (2009), 075002

[24] C. Mueller, L. Mytnik, J. Quastel: Effect of noise on front propagation in reaction-diffusion equations of KPP type. *Invent. Math.* **184** (2011), 405–453

[25] J. Nolen: A central limit theorem for pulled fronts in a random medium. *Netw. Heterog. Media* **6** (2011), 167–194

[26] D. Panja. *Effects of Fluctuations on Propagating Fronts*, Physics Reports **393** (2004), 87-174.

[27] S. Pal, J. Pitman. One-dimensional Brownian particle systems with rank dependent drifts. *Ann. Appl. Probability* **18** (2008), 2179–2207

[28] A. Ruzmaikina and M. Aizenman: Characterization of invariant measures at the leading edge for competing particle systems *Ann. Probab.* **33** (2005), 82–113

(Francis Comets) Université Paris Diderot - Paris 7, Mathématiques, case 7012, F-75 205 Paris Cedex 13, France
E-mail address: `comets@math.univ-paris-diderot.fr`

(Jeremy Quastel) Departments of Mathematics and Statistics, University of Toronto, 40 St. George Street, Toronto, Ontario M5S 1L2, Canada
E-mail address: `quastel@math.toronto.edu`

(Alejandro F. Ramírez) Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Macul, Santiago, Chile
E-mail address: `aramirez@mat.puc.cl`